# PHPE 308M/PHIL 209F
# Fairness

Eric Pacuit, University of Maryland

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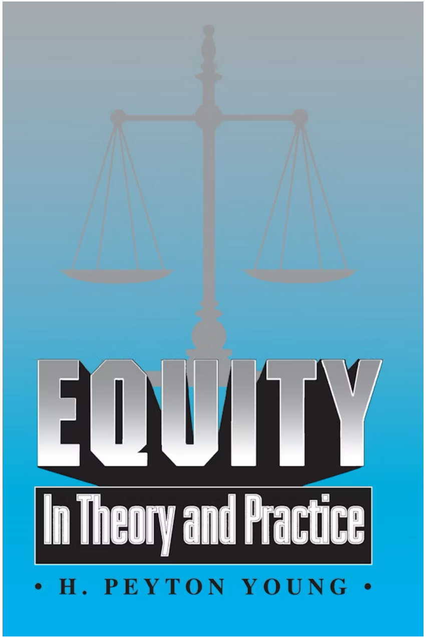
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In Theory and Practice
• H. PEYTON YOUNG •

# Envy Free Distributions

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# Envy Free Distributions

An allocation is **envy-free** when no participant would prefer to swap their allocation with any other participant's allocation.

- In other words, each person values their own portion at least as much as (or more than) they value anyone else's portion
- Note that it does not require *interpersonal comparisons of utility*, because each person evaluates every other person's portion in terms of her own utility function.

# Envy Free Distributions

$A$ might envy $B$ because $B$ is tall. To eliminate $A$'s envy requires that $A$ be made wealthier than $B$. On the other hand, $B$ might be indifferent between being tall and short. Hence if $A$ is compensated, $B$ would be envious. So there may be no way to avoid envy.

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We do not require that society in general be envy-free; we only require that no person prefer another's portion of a *particular allocation of goods*.

If an estate is being distributed among heirs, for example, the "no envy" criterion says that no heir should prefer another's portion of the property to his own. They might envy each other because of other goods that they own, or because of their different abilities and circumstances of life, but not because someone else received a more desirable portion.

# Other Fairness Constraints

- **Efficient (Pareto-Optimal)**: there is no other allocation that is at least as good for all individuals and strictly better for at least one individual.
- **Proportional**: Everyone gets at least $1/n$ of the total value (in their own estimation) when there are $n$ participants
- **Equitable**: Everyone has the **same** value of their portion

Is there any way to design an allocation procedure that leads to outcomes which are **envy-free** and **efficient**?

# Envy-Free Division

S. Brams, P. Edelman and P. Fishburn. *Paradoxes of Fair Division*. Journal of Philosophy, 98(6), pp. 300-314.

C. Klamler. *The Notion of Fair Division in Negotiations*. Handbook of Group Decision and Negotiation.

# Allocations

Suppose that $X$ is a set of items and $I$ is a set of agents, or players.

An **allocation** assigns to each agent in $I$ some of the items from $X$ such that no item is allocated to more than one agent.

- An allocation is **complete** provided that all items are allocated.
- An allocation is **balanced** provided that the agents receive the same number of items.

# Preferences

We assume that agents have preferences over the set of items $X$.

For example, if $X = \{a, b, c, d\}$, then a preference of the items might be:

$$a \succ b \succ c \succ d$$

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How can we compare subsets of these items if the only available information is the player's preference ranking of the items?

If we assume no synergies between the items, i.e., the items are neither complements nor substitutes, then we can infer some preferences between sets of items.

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- The set $\{a, b\}$ should be considered better than the set $\{c, d\}$.
- The set $\{a, c\}$ should be considered better than the set $\{b, d\}$.

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- However, it is not clear how to compare $\{b, c\}$ to $\{a, d\}$.

A player prefers one set $S$ of items to a different set $T$ if (i) $S$ has as many items as $T$ and (ii) for every item $t$ in $T$ and not in $S$, there is a distinct item $s$ in $S$ and not $T$ that the player prefers to $t$.

# Preferences

| $\succ$ | $u$ | $u'$ |
|---|---|---|
| $a$ | 10 | 10 |
| $b$ | 8 | 7 |
| $c$ | 6 | 2 |
| $d$ | 1 | 1 |

- The set $\{a, b\}$ should be preferred to the set $\{c, d\}$

$$u(\{a,b\}) = u(a) + u(b) = 18 \quad > \quad u(\{c,d\}) = u(c) + u(d) = 7$$
$$u'(\{a,b\}) = u'(a) + u'(b) = 17 \quad > \quad u'(\{c,d\}) = u'(c) + u'(d) = 3$$

# Preferences

| $\succ$ | $u$ | $u'$ |
|---|---|---|
| $a$ | 10 | 10 |
| $b$ | 8 | 7 |
| $c$ | 6 | 2 |
| $d$ | 1 | 1 |

- The set $\{a, c\}$ should be preferred to the set $\{b, d\}$

$$u(\{a, c\}) = u(a) + u(c) = 16 \quad > \quad u(\{b, d\}) = u(b) + u(d) = 9$$
$$u'(\{a, c\}) = u'(a) + u'(c) = 12 \quad > \quad u'(\{b, d\}) = u'(b) + u'(d) = 8$$

# Preferences

| $\succ$ | $u$ | $u'$ |
|---|---|---|
| $a$ | 10 | 10 |
| $b$ | 8 | 7 |
| $c$ | 6 | 2 |
| $d$ | 1 | 1 |

- However, it is not clear how to compare $\{b, c\}$ to $\{a, d\}$.

$$u(\{b, c\}) = u(b) + u(c) = 14 \quad > \quad u(\{a, d\}) = u(a) + u(d) = 11$$
$$u'(\{b, c\}) = u'(b) + u'(c) = 9 \quad < \quad u'(\{a, d\}) = u'(a) + u'(d) = 11$$

# Preferences

1. Players cannot compensate each other with side payments—the division is only of the indivisible items.

2. All players have positive values for every item.

3. A player prefers one set $S$ of items to a different set $T$ if (i) $S$ has as many items as $T$ and (ii) for every item $t$ in $T$ and not in $S$, there is a distinct item $s$ in $S$ and not $T$ that the player prefers to $t$.

| $A$ | $B$ |
|---|---|
| $a$ | $b$ |
| $b$ | $c$ |
| $c$ | $d$ |
| $d$ | $a$ |
| $e$ | $f$ |
| $f$ | $e$ |

Is there a complete envy-free division?

| $A$ | $B$ |
| --- | --- |
| $a$ | $b$ |
| $b$ | $c$ |
| $c$ | $d$ |
| $d$ | $a$ |
| $e$ | $f$ |
| $f$ | $e$ |

A complete envy-free division exists:

$A$ receives $a$, $c$, $e$
$B$ receives $b$, $d$, $f$

| $A$ | $B$ |
|---|---|
| $a$ | $b$ |
| $b$ | $c$ |
| $c$ | $d$ |
| $d$ | $e$ |
| $e$ | $a$ |
| $f$ | $f$ |

Is there any complete envy-free division?

| $A$ | $B$ |
|---|---|
| $a$ | $b$ |
| $b$ | $c$ |
| $c$ | $d$ |
| $d$ | $e$ |
| $e$ | $a$ |
| $f$ | $f$ |

There is no envy-free complete division since one of person must receive item $f$.

| $A$ | $B$ |
|---|---|
| $a$ | $b$ |
| $b$ | $c$ |
| $c$ | $a$ |
| $d$ | $f$ |
| $e$ | $e$ |
| $f$ | $d$ |

Is there a complete envy-free division?

| $A$ | $B$ |
| --- | --- |
| $a$ | $b$ |
| $b$ | $c$ |
| $c$ | $a$ |
| $d$ | $f$ |
| $e$ | $e$ |
| $f$ | $d$ |

There is no envy-free division since either $A$ or $B$ has to get at least two out of the top three items (which are the same for both agents).

| $A$ | $B$ | $C$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $c$ | $a$ |
| $c$ | $b$ | $c$ |

Is there an envy-free division?

| $A$ | $B$ | $C$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $c$ | $a$ |
| $c$ | $b$ | $c$ |

The efficient divisions $(1, 3, 2)$, $(2, 1, 3)$, and $(3, 1, 2)$ are not envy-free.
The inefficient divisions $(1, 2, 3)$, $(2, 3, 1)$, and $(3, 2, 1)$ are also not envy-free.

| | | | | | |
|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 |
| Bob: | 1 | $\succ$ | 3 | $\succ$ | 2 |
| Cath: | 2 | $\succ$ | 1 | $\succ$ | 2 |

*There are no envy-free divisions.*

# Paradoxes of Fair Division

- The conflict between efficiency and envy-freeness;
- The failure of a unique efficient and envy-free division to satisfy other fair-division criteria;
- The desirability, on occasion, of dividing items unequally.

S. Brams, P. Edelman and P. Fishburn. *Paradoxes of Fair Division*. Journal of Philosophy, 98(6), pp. 300-314.

# Envy-Freeness and Efficiency

$$\begin{array}{lccccccccccc} \text{Ann:} & 1 & \succ & 2 & \succ & 3 & \succ & 4 & \succ & 5 & \succ & 6 \\ \text{Bob:} & 4 & \succ & 3 & \succ & 2 & \succ & 1 & \succ & 5 & \succ & 6 \\ \text{Cath:} & 5 & \succ & 1 & \succ & 2 & \succ & 6 & \succ & 3 & \succ & 4 \end{array}$$

# Envy-Freeness and Efficiency

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 4 | $\succ$ | 3 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 5 | $\succ$ | 6 |
| Cath: | 5 | $\succ$ | 1 | $\succ$ | 2 | $\succ$ | 6 | $\succ$ | 3 | $\succ$ | 4 |

Ann: $\{1, 3\}$
Bob: $\{2, 4\}$
Cath: $\{5, 6\}$

# Envy-Freeness and Efficiency

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 4 | $\succ$ | 3 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 5 | $\succ$ | 6 |
| Cath: | 5 | $\succ$ | 1 | $\succ$ | 2 | $\succ$ | 6 | $\succ$ | 3 | $\succ$ | 4 |

Ann: $\{1,3\}$
Bob: $\{2,4\}$
Cath: $\{5,6\}$

Ann: $\{1,2\}$
Bob: $\{3,4\}$
Cath: $\{5,6\}$

# Envy-Freeness and Efficiency

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 4 | $\succ$ | 3 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 5 | $\succ$ | 6 |
| Cath: | 5 | $\succ$ | 1 | $\succ$ | 2 | $\succ$ | 6 | $\succ$ | 3 | $\succ$ | 4 |

| | | | |
|---|---|---|---|
| Ann: | $\{1, 3\}$ | Ann: | $\{1, 2\}$ |
| Bob: | $\{2, 4\}$ | Bob: | $\{3, 4\}$ |
| Cath: | $\{5, 6\}$ | Cath: | $\{5, 6\}$ |

# Envy-Freeness and Efficiency

Ann: $\colorbox{lightgray}{\color{red}1} \succ \colorbox{lightgray}{\color{red}2} \succ 3 \succ 4 \succ 5 \succ 6$

Bob: $4 \succ 3 \succ 2 \succ 1 \succ 5 \succ 6$

Cath: $\colorbox{lightgray}{\color{blue}5} \succ \colorbox{lightgray}{\color{red}1} \succ \colorbox{lightgray}{\color{red}2} \succ \colorbox{lightgray}{\color{blue}6} \succ 3 \succ 4$

| | |
|---|---|
| Ann: $\{1, 3\}$ | Ann: $\color{red}\{1, 2\}$ |
| Bob: $\{2, 4\}$ | Bob: $\{3, 4\}$ |
| Cath: $\{5, 6\}$ | Cath: $\color{blue}\{5, 6\}$ |

# Envy-Freeness and Efficiency

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 4 | $\succ$ | 3 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 5 | $\succ$ | 6 |
| Cath: | 5 | $\succ$ | 1 | $\succ$ | 2 | $\succ$ | 6 | $\succ$ | 3 | $\succ$ | 4 |

| | | | |
|---|---|---|---|
| Ann: | $\{1, 3\}$ | Ann: | $\{1, 2\}$ |
| Bob: | $\{2, 4\}$ | Bob: | $\{3, 4\}$ |
| Cath: | $\{5, 6\}$ | Cath: | $\{5, 6\}$ |

*There is no other division that* guarantees *envy freeness*

$$\begin{array}{lccccccccccc}
\text{Ann:} & 1 & \succ & 2 & \succ & 3 & \succ & 4 & \succ & 5 & \succ & 6 \\
\text{Bob:} & 5 & \succ & 6 & \succ & 2 & \succ & 1 & \succ & 4 & \succ & 3 \\
\text{Cath:} & 3 & \succ & 6 & \succ & 5 & \succ & 4 & \succ & 1 & \succ & 2
\end{array}$$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 5 | $\succ$ | 6 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | 3 | $\succ$ | 6 | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

- Three efficient divisions: $(12, 56, 34)$, $(12, 45, 36)$ and $(14, 25, 36)$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | **1** | $\succ$ | **2** | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | **5** | $\succ$ | 6 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | **4** | $\succ$ | 3 |
| Cath: | **3** | $\succ$ | **6** | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

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| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 5 | $\succ$ | 6 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | 3 | $\succ$ | 6 | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

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| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 5 | $\succ$ | 6 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | 3 | $\succ$ | 6 | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

- Three efficient divisions: $(12, 56, 34)$, $(12, 45, 36)$ and $(14, 25, 36)$
- The only envy-free and efficient division is $(14, 25, 36)$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 5 | $\succ$ | 6 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | 3 | $\succ$ | 6 | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

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| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 5 | $\succ$ | 6 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | 3 | $\succ$ | 6 | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

- Three efficient divisions: $(12, 56, 34)$, $(12, 45, 36)$ and $(14, 25, 36)$
- The only envy-free and efficient division is $(14, 25, 36)$

# Voting

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | **1** | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | **4** | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | **5** | $\succ$ | 6 | $\succ$ | **2** | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | **3** | $\succ$ | **6** | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

| Allocations | Preferences |
|---|---|
| $A_1$: $(12, 56, 34)$ | Ann: $A_1 \; I_A \; A_2 \; P_A \; A_3$ |
| $A_2$: $(12, 45, 36)$ | Bob: $A_1 \; P_B \; A_3 \; P_B \; A_2$ |
| $A_3$: $(14, 25, 36)$ | Cath: $A_2 \; I_C \; A_3 \; P_C \; A_1$ |

# Voting

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | **1** | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | **4** | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | **5** | $\succ$ | 6 | $\succ$ | **2** | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | **3** | $\succ$ | **6** | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

| Allocations | Preferences |
|---|---|
| $A_1$: $(12, 56, 34)$ | Ann: $A_1 \; I_A \; A_2 \; P_A \; A_3$ |
| $A_2$: $(12, 45, 36)$ | Bob: $A_1 \; P_B \; A_3 \; P_B \; A_2$ |
| $A_3$: $(14, 25, 36)$ | Cath: $A_2 \; I_C \; A_3 \; P_C \; A_1$ |

**Conclusion**: The unique envy-free division would lose in a vote to any of the other efficient divisions

# Maximize Total Utility

| Utility | 6 | | 5 | | 4 | | 3 | | 2 | | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | **1** | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | **4** | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | **5** | $\succ$ | 6 | $\succ$ | **2** | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | **3** | $\succ$ | **6** | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

| Allocations | Total Utility |
|---|---|
| $A_1$: $(12, 56, 34)$ | 31 |
| $A_2$: $(12, 45, 36)$ | 30 |
| $A_3$: $(14, 25, 36)$ | 30 |

# Maximize Total Utility

| Utility | 6 | | 5 | | 4 | | 3 | | 2 | | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 5 | $\succ$ | 6 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | 3 | $\succ$ | 6 | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

| Allocations | Total Utility |
|---|---|
| $A_1$: $(12, 56, 34)$ | 31 |
| $A_2$: $(12, 45, 36)$ | 30 |
| $A_3$: $(14, 25, 36)$ | 30 |

**Conclusion**: Maximizing the total utility (i.e., the modified Borda score) will not select the unique envy-free division.

# Improve the Worse Off

| Utility | 6 | | 5 | | 4 | | 3 | | 2 | | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | **1** | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | **4** | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | **5** | $\succ$ | 6 | $\succ$ | **2** | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | **3** | $\succ$ | **6** | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

| Allocations | Minimum Utilities |
|---|---|
| $A_1$: $(12, 56, 34)$ | (5, 5, 3) |
| $A_2$: $(12, 45, 36)$ | (5, 2, 5) |
| $A_3$: $(14, 25, 36)$ | (3, 4, 5) |

# Improve the Worse Off

| Utility | 6 | | 5 | | 4 | | 3 | | 2 | | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Ann: | 1 | $\succ$ | 2 | $\succ$ | 3 | $\succ$ | 4 | $\succ$ | 5 | $\succ$ | 6 |
| Bob: | 5 | $\succ$ | 6 | $\succ$ | 2 | $\succ$ | 1 | $\succ$ | 4 | $\succ$ | 3 |
| Cath: | 3 | $\succ$ | 6 | $\succ$ | 5 | $\succ$ | 4 | $\succ$ | 1 | $\succ$ | 2 |

| Allocations | Minimum Utilities |
|---|---|
| $A_1$: $(12, 56, 34)$ | (5, 5, 3) |
| $A_2$: $(12, 45, 36)$ | (5, 2, 5) |
| $A_3$: $(14, 25, 36)$ | (3, 4, 5) |

**Conclusion**: (Lexicographic) Maximin will not select the unique envy-free division.