# PHIL 408Q/PHPE 308D Fairness

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S. Brams, P. Edelman and P. Fishburn. *Paradoxes of Fair Division*. Journal of Philosophy, 98(6), pp. 300-314.

C. Klamler. *The Notion of Fair Division in Negotiations* . Handbook of Group Decision and Negotiation.

Suppose that X is a set of items and I is a set of agents, or players.

An **allocation** assigns to each agent in I some of the items from X such that no item is allocated to more than one agent.

- > An allocation is **complete** provided that all items are allocated.
- An allocation is **balanced** provided that the agents receive the same number of items.

We assume that agents have preferences over the set of items X.

For example, if  $X = \{a, b, c, d\}$ , then a preference of the items might be:

 $a \succ b \succ c \succ d$ 

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How can we compare subsets of these items if the only available information is the player's preference ranking of the items?

If we assume no synergies between the items, i.e., the items are neither complements nor substitutes, then we can infer some preferences between sets of items.

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The set {a, b} should be considered better than the set {c, d}.
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• The set  $\{a, b\}$  should be considered better than the set  $\{c, d\}$ .

- The set  $\{a, c\}$  should be considered better than the set  $\{b, d\}$ .
- However, it is not clear how to compare  $\{b, c\}$  to  $\{a, d\}$ .

A player prefers one set S of items to a different set T if (i) S has as many items as T and (ii) for every item t in T and not in S, there is a distinct item s in S and not T that the player prefers to t.

$\succ$	u	u'
а	10	10
Ь	8	7
С	6	2
d	1	1

• The set  $\{a, b\}$  should be preferred to the set  $\{c, d\}$ 

$$u(\{a,b\}) = u(a) + u(b) = 18 > u(\{c,d\}) = u(c) + u(d) = 7$$
  
$$u'(\{a,b\}) = u'(a) + u'(b) = 17 > u'(\{c,d\}) = u'(c) + u'(d) = 3$$

$\succ$	u	u'
а	10	10
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С	6	2
d	1	1

• The set  $\{a, c\}$  should be preferred to the set  $\{b, d\}$ 

$$u(\{a,c\}) = u(a) + u(c) = 16 > u(\{b,d\}) = u(b) + u(d) = 9$$
  
$$u'(\{a,c\}) = u'(a) + u'(c) = 12 > u'(\{b,d\}) = u'(b) + u'(d) = 8$$

$\succ$	u	u'
а	10	10
Ь	8	7
С	6	2
d	1	1

• However, it is not clear how to compare  $\{b, c\}$  to  $\{a, d\}$ .

$$u(\{b,c\}) = u(b) + u(c) = 14 > u(\{a,d\}) = u(a) + u(d) = 11$$
  
$$u'(\{b,c\}) = u'(b) + u'(c) = 9 < u'(\{a,d\}) = u'(a) + u'(d) = 11$$

1. Players cannot compensate each other with side payments—the division is only of the indivisible items.

2. All players have positive values for every item.

3. A player prefers one set S of items to a different set T if (i) S has as many items as T and (ii) for every item t in T and not in S, there is a distinct item s in S and not T that the player prefers to t.

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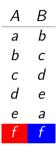
## Fairness Conditions

Envy-Free: each player weakly prefers her own set of items to the other player's set of items. This ensures that there is no pressure on the players to swap their sets of items with other players and guarantees a certain kind of stability.

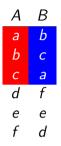
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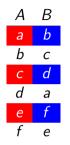
Efficiency (Pareto-Optimality): there is no other allocation that is at least as good for all players and strictly better for at least one player.



There is no envy-free complete division since one of person must receive item f.



There is no envy-free division since either A or B has to get at least two out of the top three items (which are the same for both agents).



A complete envy-free division exists:

A receives a, c, e B receives b, d, f

## Paradoxes of Fair Division

- The conflict between efficiency and envy-freeness;
- The failure of a unique efficient and envy-free division to satisfy other fair-division criteria;
- > The desirability, on occasion, of dividing items unequally.

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Ann:1
$$\succ$$
2 $\succ$ 3 $\succ$ 4 $\succ$ 5 $\succ$ 6Bob:4 $\succ$ 3 $\succ$ 2 $\succ$ 1 $\succ$ 5 $\succ$ 6Cath:5 $\succ$ 1 $\succ$ 2 $\succ$ 6 $\succ$ 3 $\succ$ 4

Ann:1
$$\succ$$
2 $\succ$ 3 $\succ$ 4 $\succ$ 5 $\succ$ 6Bob:4 $\succ$ 3 $\succ$ 2 $\succ$ 1 $\succ$ 5 $\succ$ 6Cath:5 $\succ$ 1 $\succ$ 2 $\succ$ 6 $\succ$ 3 $\succ$ 4

Ann:  $\{1, 3\}$ Bob:  $\{2, 4\}$ Cath:  $\{5, 6\}$ 

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$$\succ$$
2 $\succ$ 3 $\succ$ 4 $\succ$ 5 $\succ$ 6Bob:4 $\succ$ 3 $\succ$ 2 $\succ$ 1 $\succ$ 5 $\succ$ 6Cath:5 $\succ$ 1 $\succ$ 2 $\succ$ 6 $\succ$ 3 $\succ$ 4

Ann: {1,3}	Ann: {1,2}
Bob: {2,4}	Bob: {3, 4}
Cath: $\{5, 6\}$	Cath: $\{5, 6\}$

Ann:1
$$\succ$$
2 $\succ$ 3 $\succ$ 4 $\succ$ 5 $\succ$ 6Bob:4 $\succ$ 3 $\succ$ 2 $\succ$ 1 $\succ$ 5 $\succ$ 6Cath:5 $\succ$ 1 $\succ$ 2 $\succ$ 6 $\succ$ 3 $\succ$ 4



Ann: {1,3}	Ann: {1,2}
Bob: {2,4}	Bob: {3,4}
Cath: $\{5, 6\}$	Cath: {5, 6}

There is no other division that guarantees envy freeness

## No Envy-Free Division

Ann:  $1 \succ 2 \succ 3$ Bob:  $1 \succ 3 \succ 2$ Cath:  $2 \succ 1 \succ 2$ 

## No Envy-Free Division

There are no envy-free divisions.

Ann:1
$$\succ$$
2 $\succ$ 3 $\succ$ 4 $\succ$ 5 $\succ$ 6Bob:5 $\succ$ 6 $\succ$ 2 $\succ$ 1 $\succ$ 4 $\succ$ 3Cath:3 $\succ$ 6 $\succ$ 5 $\succ$ 4 $\succ$ 1 $\succ$ 2

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▶ Three efficient divisions: (12, 56, 34), (12, 45, 36) and (14, 25, 36)

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# Voting

Ann:1
$$\succ$$
2 $\succ$ 3 $\succ$ 4 $\succ$ 5 $\succ$ 6Bob:5 $\succ$ 6 $\succ$ 2 $\succ$ 1 $\succ$ 4 $\succ$ 3Cath:3 $\succ$ 6 $\succ$ 5 $\succ$ 4 $\succ$ 1 $\succ$ 2

<u>Allocations</u>	Preferences						
<i>A</i> <sub>1</sub> : (12, 56, 34)	Ann: $A_1 I_A A_2 P_A A_3$						
<i>A</i> <sub>2</sub> : (12, 45, 36)	Bob: $A_1 P_B A_3 P_B A_2$						
<i>A</i> <sub>3</sub> : (14, 25, 36)	Cath: $A_2 I_C A_3 P_C A_1$						

# Voting

Ann:1
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2 $\succ$ 3 $\succ$ 4 $\succ$ 5 $\succ$ 6Bob:5 $\succ$ 6 $\succ$ 2 $\succ$ 1 $\succ$ 4 $\succ$ 3Cath:3 $\succ$ 6 $\succ$ 5 $\succ$ 4 $\succ$ 1 $\succ$ 2

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<i>A</i> <sub>3</sub> : (14, 25, 36)	Cath: $A_2 I_C A_3 P_C A_1$

**Conclusion**: The unique envy-free division would lose in a vote to any of the other efficient divisions

# Maximize Total Utility

Utility	6		5		4		3		2		1	
Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6	
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3	
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2	
Allocations								Total Utility				
		$A_1$ :	(12)	2, 56	, 34	)		31	-			
		$A_2$ :	(12)	2, 45	, 36	)		30	)			
		<i>A</i> <sub>3</sub> :	(1	4, 25	, 36	)		30	)			

## Maximize Total Utility

Utility	6		5		4		3		2		1
Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2
Allocations								tal l	Jtilit	сy	
		$A_1$ :	(12)	2, 56	, 34	)		31			
		$A_2$ :	(12)	2, 45	, 36	)		30	)		
		<i>A</i> <sub>3</sub> :	(1	4, 25	, 36	)		30	)		

**Conclusion**: Maximizing the total utility (i.e., the modified Borda score) will not select the unique envy-free division.

## Improve the Worse Off

Utility	6		5		4		3		2		1
Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2
		A	Alloc	catio	ns		Mi	nimı	ım l	Utilit	ies
		$A_1$ :	(12)	2, 56	, 34	)		(5	, 5,	3)	
		$A_2: (12, 45, 36) (5, 2, 5)$									
		<i>A</i> <sub>3</sub> :	(14)	4, 25	, 36	)	(3, 4, 5)				

## Improve the Worse Off

Utility	6		5		4		3		2		1
Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2
Allocations Minimum Utilities											ies
		$A_1$ :	(12)	2, 56	, 34	)		(5	, 5,	3)	
		$A_2$ : (12, 45, 36) (5, 2, 5)									
		<i>A</i> <sub>3</sub> :	(14	4, 25	, 36	)		(3	, 4,	5)	

**Conclusion**: (Lexicographic) Maximin will not select the unique envy-free division.