PHIL 408Q/PHPE 308D
Fairness

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March 5, 2024

Envy-Free Division

S. Brams, P. Edelman and P. Fishburn. *Paradoxes of Fair Division*. Journal of Philosophy, 98(6), pp. 300-314.

C. Klamler. *The Notion of Fair Division in Negotiations* . Handbook of Group Decision and Negotiation.

Allocations

Suppose that X is a set of items and I is a set of agents, or players.

An **allocation** assigns to each agent in I some of the items from X such that no item is allocated to more than one agent.

- An allocation is **complete** provided that all items are allocated.
- An allocation is **balanced** provided that the agents receive the same number of items.

Preferences

We assume that agents have preferences over the set of items X.

For example, if $X = \{a, b, c, d\}$, then a preference of the items might be:

$$a \succ b \succ c \succ d$$

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How can we compare subsets of these items if the only available information is the player's preference ranking of the items?

If we assume no synergies between the items, i.e., the items are neither complements nor substitutes, then we can infer some preferences between sets of items.

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We assume that agents have preferences over the set of items X.

For example, if $X = \{a, b, c, d\}$, then a preference of the items might be:

$$a \succ b \succ c \succ d$$

- The set $\{a, b\}$ should be considered better than the set $\{c, d\}$.
- The set $\{a, c\}$ should be considered better than the set $\{b, d\}$.

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- The set $\{a, c\}$ should be considered better than the set $\{b, d\}$.
- However, it is not clear how to compare $\{b, c\}$ to $\{a, d\}$.

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- The set $\{a, c\}$ should be considered better than the set $\{b, d\}$.
- However, it is not clear how to compare $\{b, c\}$ to $\{a, d\}$.

A player prefers one set S of items to a different set T if (i) S has as many items as T and (ii) for every item t in T and not in S, there is a distinct item s in S and not T that the player prefers to t.

Preferences

$\succ$	u	u'
a	10	10
b	8	7
c	6	2
d	1	1

- The set $\{a, b\}$ should be preferred to the set $\{c, d\}$

$$u(\{a,b\}) = u(a) + u(b) = 18 \quad > \quad u(\{c,d\}) = u(c) + u(d) = 7$$
$$u'(\{a,b\}) = u'(a) + u'(b) = 17 \quad > \quad u'(\{c,d\}) = u'(c) + u'(d) = 3$$

Preferences

$\succ$	u	u'
a	10	10
b	8	7
c	6	2
d	1	1

- The set $\{a, c\}$ should be preferred to the set $\{b, d\}$

$$u(\{a, c\}) = u(a) + u(c) = 16 \quad > \quad u(\{b, d\}) = u(b) + u(d) = 9$$
$$u'(\{a, c\}) = u'(a) + u'(c) = 12 \quad > \quad u'(\{b, d\}) = u'(b) + u'(d) = 8$$

Preferences

$\succ$	u	u'
a	10	10
b	8	7
c	6	2
d	1	1

- However, it is not clear how to compare $\{b, c\}$ to $\{a, d\}$.

$$u(\{b,c\}) = u(b) + u(c) = 14 \quad > \quad u(\{a,d\}) = u(a) + u(d) = 11$$
$$u'(\{b,c\}) = u'(b) + u'(c) = 9 \quad < \quad u'(\{a,d\}) = u'(a) + u'(d) = 11$$

Preferences

1. Players cannot compensate each other with side payments—the division is only of the indivisible items.

2. All players have positive values for every item.

3. A player prefers one set S of items to a different set T if (i) S has as many items as T and (ii) for every item t in T and not in S, there is a distinct item s in S and not T that the player prefers to t.

Preferences

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Fairness Conditions

- **Envy-Free:** each player weakly prefers her own set of items to the other player's set of items. This ensures that there is no pressure on the players to swap their sets of items with other players and guarantees a certain kind of stability.

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- **Envy-Free:** each player weakly prefers her own set of items to the other player's set of items. This ensures that there is no pressure on the players to swap their sets of items with other players and guarantees a certain kind of stability.

- **Efficiency (Pareto-Optimality):** there is no other allocation that is at least as good for all players and strictly better for at least one player.

A	B
a	b
b	c
c	d
d	e
e	a
f	f

There is no envy-free complete division since one of person must receive item f.

A	B
a	b
b	c
c	a
d	f
e	e
f	d

There is no envy-free division since either A or B has to get at least two out of the top three items (which are the same for both agents).

A	B
a	b
b	c
c	d
d	a
e	f
f	e

A complete envy-free division exists:

A receives a, c, e
B receives b, d, f

Paradoxes of Fair Division

- The conflict between efficiency and envy-freeness;
- The failure of a unique efficient and envy-free division to satisfy other fair-division criteria;
- The desirability, on occasion, of dividing items unequally.

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Envy-Freeness and Efficiency

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	4	$\succ$	3	$\succ$	2	$\succ$	1	$\succ$	5	$\succ$	6
Cath:	5	$\succ$	1	$\succ$	2	$\succ$	6	$\succ$	3	$\succ$	4

Envy-Freeness and Efficiency

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	4	$\succ$	3	$\succ$	2	$\succ$	1	$\succ$	5	$\succ$	6
Cath:	5	$\succ$	1	$\succ$	2	$\succ$	6	$\succ$	3	$\succ$	4

Ann: $\{1, 3\}$
Bob: $\{2, 4\}$
Cath: $\{5, 6\}$

Envy-Freeness and Efficiency

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	4	$\succ$	3	$\succ$	2	$\succ$	1	$\succ$	5	$\succ$	6
Cath:	5	$\succ$	1	$\succ$	2	$\succ$	6	$\succ$	3	$\succ$	4

Ann: $\{1,3\}$	Ann: $\{1,2\}$
Bob: $\{2,4\}$	Bob: $\{3,4\}$
Cath: $\{5,6\}$	Cath: $\{5,6\}$

Envy-Freeness and Efficiency

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	4	$\succ$	3	$\succ$	2	$\succ$	1	$\succ$	5	$\succ$	6
Cath:	5	$\succ$	1	$\succ$	2	$\succ$	6	$\succ$	3	$\succ$	4

Ann:	$\{1, 3\}$	Ann:	$\{1, 2\}$
Bob:	$\{2, 4\}$	Bob:	$\{3, 4\}$
Cath:	$\{5, 6\}$	Cath:	$\{5, 6\}$

Envy-Freeness and Efficiency

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	4	$\succ$	3	$\succ$	2	$\succ$	1	$\succ$	5	$\succ$	6
Cath:	5	$\succ$	1	$\succ$	2	$\succ$	6	$\succ$	3	$\succ$	4

Ann:	$\{1, 3\}$	Ann:	$\{1, 2\}$
Bob:	$\{2, 4\}$	Bob:	$\{3, 4\}$
Cath:	$\{5, 6\}$	Cath:	$\{5, 6\}$

There is no other division that guarantees *envy freeness*

No Envy-Free Division

Ann:	1	$\succ$	2	$\succ$	3
Bob:	1	$\succ$	3	$\succ$	2
Cath:	2	$\succ$	1	$\succ$	2

No Envy-Free Division

Ann:	1	$\succ$	2	$\succ$	3
Bob:	1	$\succ$	3	$\succ$	2
Cath:	2	$\succ$	1	$\succ$	2

There are no envy-free divisions.

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

- Three efficient divisions: $(12, 56, 34)$, $(12, 45, 36)$ and $(14, 25, 36)$

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

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Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

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Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

- Three efficient divisions: $(12, 56, 34)$, $(12, 45, 36)$ and $(14, 25, 36)$
- The only envy-free and efficient division is $(14, 25, 36)$

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

- Three efficient divisions: $(12, 56, 34)$, $(12, 45, 36)$ and $(14, 25, 36)$
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Ann:	**1**	$\succ$	2	$\succ$	3	$\succ$	**4**	$\succ$	5	$\succ$	6
Bob:	**5**	$\succ$	6	$\succ$	**2**	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	**3**	$\succ$	**6**	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

- Three efficient divisions: $(12, 56, 34)$, $(12, 45, 36)$ and $(14, 25, 36)$
- The only envy-free and efficient division is $(14, 25, 36)$

Voting

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

Allocations	Preferences
A_1: $(12, 56, 34)$	Ann: $A_1 \; I_A \; A_2 \; P_A \; A_3$
A_2: $(12, 45, 36)$	Bob: $A_1 \; P_B \; A_3 \; P_B \; A_2$
A_3: $(14, 25, 36)$	Cath: $A_2 \; I_C \; A_3 \; P_C \; A_1$

Voting

Ann:	**1**	$\succ$	2	$\succ$	3	$\succ$	**4**	$\succ$	5	$\succ$	6
Bob:	**5**	$\succ$	6	$\succ$	**2**	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	**3**	$\succ$	**6**	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

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A_1: $(12, 56, 34)$	Ann: $A_1 \; I_A \; A_2 \; P_A \; A_3$
A_2: $(12, 45, 36)$	Bob: $A_1 \; P_B \; A_3 \; P_B \; A_2$
A_3: $(14, 25, 36)$	Cath: $A_2 \; I_C \; A_3 \; P_C \; A_1$

Conclusion: The unique envy-free division would lose in a vote to any of the other efficient divisions

Maximize Total Utility

Utility	6		5		4		3		2		1
Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

Allocations	Total Utility
A_1: $(12, 56, 34)$	31
A_2: $(12, 45, 36)$	30
A_3: $(14, 25, 36)$	30

Maximize Total Utility

Utility	6		5		4		3		2		1
Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

Allocations	Total Utility
A_1: $(12, 56, 34)$	31
A_2: $(12, 45, 36)$	30
A_3: $(14, 25, 36)$	30

Conclusion: Maximizing the total utility (i.e., the modified Borda score) will not select the unique envy-free division.

Improve the Worse Off

Utility	6		5		4		3		2		1
Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	5	$\succ$	6	$\succ$	2	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	3	$\succ$	6	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

Allocations	Minimum Utilities
A_1: $(12, 56, 34)$	(5, 5, 3)
A_2: $(12, 45, 36)$	(5, 2, 5)
A_3: $(14, 25, 36)$	(3, 4, 5)

Improve the Worse Off

Utility	6		5		4		3		2		1
Ann:	**1**	$\succ$	2	$\succ$	3	$\succ$	**4**	$\succ$	5	$\succ$	6
Bob:	**5**	$\succ$	6	$\succ$	**2**	$\succ$	1	$\succ$	4	$\succ$	3
Cath:	**3**	$\succ$	**6**	$\succ$	5	$\succ$	4	$\succ$	1	$\succ$	2

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A_1: $(12, 56, 34)$	(5, 5, 3)
A_2: $(12, 45, 36)$	(5, 2, 5)
A_3: $(14, 25, 36)$	(3, 4, 5)

Conclusion: (Lexicographic) Maximin will not select the unique envy-free division.