PHIL 408Q/PHPE 308D
Fairness

Eric Pacuit, University of Maryland

March 12, 2024

R. Sugden (1984). *Is Fairness Good? A Critique of Varian's Theory of Fairness*. Noûs, 18(3), pp. 505-511.

Envy: to envy someone is not to feel ill-will towards him, or to experience disutility when reflecting on his good fortune; it is simply to prefer what he has to what one has oneself. An allocation is **envy-free** if no person envies any other person.

Pareto-efficient: no other feasible allocation exists such that at least one person prefers the latter to the former and no one prefers the former to the latter.

If an Pareto-efficient and envy-free allocation is feasible, then no allocation that is not Pareto-efficient and envy-free should be chosen.

If an Pareto-efficient and envy-free allocation is feasible, then no allocation that is not Pareto-efficient and envy-free should be chosen.

What is the justification for this principle?

Varian on Fairness

Pareto-efficiency: ‘the criterion of Pareto efficiency is certainly reasonable: if there is some way to make everyone better off, why not do it?’.

Varian on Fairness

Pareto-efficiency: ‘the criterion of Pareto efficiency is certainly reasonable: if there is some way to make everyone better off, why not do it?’.

A second claim is that envy-free allocations ensure ‘equity’:

1. The definition treats all persons symmetrically
2. The definition of *envy-freeness* is just a formal definition that is not meant to reflect ordinary usage.
3. *Envy-freeness* ‘is of interest in formalizing certain ordinary concepts of equity’.

H. R. Varian (1975). *Distributive Justice, Welfare Economics, and the Theory of Fairness*. Philosophy and Public Affairs 4(1975), pp. 223-247.

But what follows is not a moral argument, nor an analysis of how people ordinarily use the concept of equity. It is a technical argument, designed to show that this concept of fairness is ‘quite operational.’

But what follows is not a moral argument, nor an analysis of how people ordinarily use the concept of equity. It is a technical argument, designed to show that this concept of fairness is ‘quite operational.’ Varian means that it requires no more information than is contained in individuals’ preference orderings; in particular, it does not require any interpersonal comparisons of utility.

But what follows is not a moral argument, nor an analysis of how people ordinarily use the concept of equity. It is a technical argument, designed to show that this concept of fairness is ‘quite operational.’ Varian means that it requires no more information than is contained in individuals’ preference orderings; in particular, it does not require any interpersonal comparisons of utility.

This may provide an answer to the practical question, ‘Could this concept of fairness be used to guide social choices?’; but Varian never answers the equally important moral question, ‘Why should it be so used?’

But what follows is not a moral argument, nor an analysis of how people ordinarily use the concept of equity. It is a technical argument, designed to show that this concept of fairness is 'quite operational.' Varian means that it requires no more information than is contained in individuals' preference orderings; in particular, it does not require any interpersonal comparisons of utility.

This may provide an answer to the practical question, 'Could this concept of fairness be used to guide social choices?'; but Varian never answers the equally important moral question, 'Why should it be so used?'

He seems to rely on his readers sharing his intuitions that Pareto-efficiency is good and that envy-freeness is equivalent to equity.

Paretian Welfare

1. To the extent that we are concerned with a person's welfare, we must concern ourselves only with what he *wants* (rather than, for example, with what we think is good for him).
2. The welfare of society depends only on the welfare of the individuals who comprise it.
3. If one person's welfare increases, other things remaining equal, then social welfare increases....'social welfare' is, in effect, being used as a synonym for 'the good of society, all things considered.'

This can be reduced to a single maxim: *as far as social choice is concerned, all that matters is the satisfaction of wants*.

Pareto inefficient outcomes are *dominated* along the n-dimensions of the individuals ‘wants’.

			$\cdots$		$\cdots$	
A	1	10	$\cdots$	7	$\cdots$	6
			$\cdots$		$\cdots$	
B	1	10	$\cdots$	5	$\cdots$	3
			$\cdots$		$\cdots$	
Individuals	1	2	$\cdots$	k	$\cdots$	n

Pareto inefficient outcomes are *dominated* along the n-dimensions of the individuals ‘wants’.

			$\cdots$		$\cdots$	
A	1	10	$\cdots$	7	$\cdots$	6
			$\cdots$		$\cdots$	
B	1	10	$\cdots$	5	$\cdots$	3
			$\cdots$		$\cdots$	
Individuals	1	2	$\cdots$	k	$\cdots$	n

Inefficient allocations ought not to be chosen: Do not choose the allocation B since A is clearly superior in the sense that it is more effective of satisfying the wants of the n individuals.

There is nothing in the Paretian theory that implies that all efficient allocations are equally good, or that every efficient allocation is better than every inefficient one.

There is nothing in the Paretian theory that implies that all efficient allocations are equally good, or that every efficient allocation is better than every inefficient one.

Two different rules:

- ✗ Choose any allocation so long as it is Pareto-efficient.
- ✓ Don't choose a Pareto-inefficient allocation.

The second rule is the rule that Varian wants to call 'certainly reasonable.'

But Varian also wants to say that envy-freeness is good.

But Varian also wants to say that envy-freeness is good. Here he runs into a problem.

But Varian also wants to say that envy-freeness is good. Here he runs into a problem.

- The rule of not choosing Pareto-inefficient allocations is only as reasonable as the principle from which it derives-that nothing matters except the satisfaction of wants.

But Varian also wants to say that envy-freeness is good. Here he runs into a problem.

- The rule of not choosing Pareto-inefficient allocations is only as reasonable as the principle from which it derives-that nothing matters except the satisfaction of wants.
- But if nothing matters except the satisfaction of wants, there seems to be no reason for valuing envy-freeness.

- To say that I envy you is to say that I prefer your bundle of goods to my own-that I want your bundle more than I want mine.

- To say that I envy you is to say that I prefer your bundle of goods to my own-that I want your bundle more than I want mine.

- Thus envy cannot exist unless there are some unsatisfied wants.

- To say that I envy you is to say that I prefer your bundle of goods to my own-that I want your bundle more than I want mine.
- Thus envy cannot exist unless there are some unsatisfied wants.
- But the absence of envy does not entail that all wants are satisfied; and eliminating envy is not the same thing as satisfying wants.

- To say that I envy you is to say that I prefer your bundle of goods to my own-that I want your bundle more than I want mine.
- Thus envy cannot exist unless there are some unsatisfied wants.
- But the absence of envy does not entail that all wants are satisfied; and eliminating envy is not the same thing as satisfying wants.
    - If you have cream with your pie while I don't, I envy you. If instead we both go without cream, there is no envy.
    - But I want cream just as much in either case; your forgoing it eliminates my envy but it does nothing to satisfy my want.

- To say that I envy you is to say that I prefer your bundle of goods to my own-that I want your bundle more than I want mine.
- Thus envy cannot exist unless there are some unsatisfied wants.
- But the absence of envy does not entail that all wants are satisfied; and eliminating envy is not the same thing as satisfying wants.
  - If you have cream with your pie while I don't, I envy you. If instead we both go without cream, there is no envy.
  - But I want cream just as much in either case; your forgoing it eliminates my envy but it does nothing to satisfy my want.

If envy-freeness is good, it is because something matters apart from the satisfaction of wants.

The idea that envy-freeness is good seems to correspond with a certain kind of egalitarianism, which emphasizes the virtues of social harmony and fraternity.

The idea that envy-freeness is good seems to correspond with a certain kind of egalitarianism, which emphasizes the virtues of social harmony and fraternity.

A society without envy is one in which everyone is happy with his lot and in which no one regards anyone else's position as worth striving for.

The idea that envy-freeness is good seems to correspond with a certain kind of egalitarianism, which emphasizes the virtues of social harmony and fraternity.

A society without envy is one in which everyone is happy with his lot and in which no one regards anyone else's position as worth striving for.

This is not everyone's idea of a good society, but it is some people's. For such people equality or perhaps more accurately harmony-is a dimension of social welfare in its own right, and an envy-free society is one that has achieved the maximum possible degree of harmony.

The idea that envy-freeness is good seems to correspond with a certain kind of egalitarianism, which emphasizes the virtues of social harmony and fraternity.

A society without envy is one in which everyone is happy with his lot and in which no one regards anyone else's position as worth striving for.

This is not everyone's idea of a good society, but it is some people's. For such people equality or perhaps more accurately harmony-is a dimension of social welfare in its own right, and an envy-free society is one that has achieved the maximum possible degree of harmony.

This is a perfectly coherent position to take, but it is inconsistent with the Paretian theory

Varian's principle of fairness amounts, in effect, to the following rule:

> First assume that all that matters is want-satisfaction, and rule out any allocations that would be clearly inferior if that assumption were true; then assume that all that matters is harmony and rule out any allocations that would be clearly inferior if that assumption were true.

Varian's principle of fairness amounts, in effect, to the following rule:

> First assume that all that matters is want-satisfaction, and rule out any allocations that would be clearly inferior if that assumption were true; then assume that all that matters is harmony and rule out any allocations that would be clearly inferior if that assumption were true.

But since these two assumptions contradict one another, any attempt to justify a rule based on both of them seems doomed to failure.

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

Feasible allocations:

- A_1 gives us both jam roll
- A_2 gives me jam roll and you jam roll with custard
- A_3 gives us both apple pie.

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

Feasible allocations:

- A_1 gives us both jam roll
- A_2 gives me jam roll and you jam roll with custard
- A_3 gives us both apple pie.

A_2 and A_3 are Pareto-efficient but A_1 is not.

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

Feasible allocations:

- A_1 gives us both jam roll
- A_2 gives me jam roll and you jam roll with custard
- A_3 gives us both apple pie.

A_2 and A_3 are Pareto-efficient but A_1 is not.

A_1 and A_3 are envy-free but A_2 is not (I envy your custard.)

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

Feasible allocations:

- A_1 gives us both jam roll
- A_2 gives me jam roll and you jam roll with custard
- A_3 gives us both apple pie.

A_2 and A_3 are Pareto-efficient but A_1 is not.

A_1 and A_3 are envy-free but A_2 is not (I envy your custard.)

So, we must choose the allocation A_3 since it is the only envy-free and Pareto-efficient allocation.

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

But what grounds are there for saying that A_3 is better than A_1 or A_2 (both give me a jam roll)?

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

But what grounds are there for saying that A_3 is better than A_1 or A_2 (both give me a jam roll)?

The satisfaction of my wants is one dimension of social welfare, and I prefer both A_1 and A_2 to A_3. So the simple proposition that want satisfaction and harmony both matter cannot provide a sufficient reason for choosing A_3.

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

But what grounds are there for saying that A_3 is better than A_1 or A_2 (both give me a jam roll)?

The satisfaction of my wants is one dimension of social welfare, and I prefer both A_1 and A_2 to A_3. So the simple proposition that want satisfaction and harmony both matter cannot provide a sufficient reason for choosing A_3.

To justify this choice, some argument must be made about the relative importance of satisfying your want for apple pie as opposed to my want for jam roll, and this is exactly the kind of argument that Varian claims to avoid.

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

Feasible allocations:

- A_1 gives us both jam roll
- ~~A_2 gives me jam roll and you jam roll with custard~~
- A_3 gives us both apple pie.
- A_4 gives me apple pie and you apple pie with cream.

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

Feasible allocations:

- A_1 gives us both jam roll
- ~~A_2 gives me jam roll and you jam roll with custard~~
- A_3 gives us both apple pie.
- A_4 gives me apple pie and you apple pie with cream.

A_1 and A_4 are Pareto-efficient but A_3 is not.

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

Feasible allocations:

- A_1 gives us both jam roll
- ~~A_2 gives me jam roll and you jam roll with custard~~
- A_3 gives us both apple pie.
- A_4 gives me apple pie and you apple pie with cream.

A_1 and A_4 are Pareto-efficient but A_3 is not.

A_1 and A_3 are envy-free but A_4 is not (I envy your cream.)

Me	You
jam roll with custard	apple pie with cream
jam roll	apple pie
apple pie with cream	jam roll with custard
apple pie	jam roll

Feasible allocations:

- A_1 gives us both jam roll
- ~~A_2 gives me jam roll and you jam roll with custard~~
- A_3 gives us both apple pie.
- A_4 gives me apple pie and you apple pie with cream.

A_1 and A_4 are Pareto-efficient but A_3 is not.
A_1 and A_3 are envy-free but A_4 is not (I envy your cream.)
So, we must choose the allocation A_1 since it is the only envy-free and Pareto-efficient allocation.

But why is A_3 better than A_1 in one case and worse in the other?

But why is A_3 better than A_1 in one case and worse in the other? The degree to which an allocation satisfies wants and the degree to which it inspires envy are both surely independent of whether other allocations happen to be feasible or not.

Cake-Cutting Problem

Main Question

How do we cut a cake fairly?

Main Question

How do we cut a cake fairly?

- A cake is a metaphor for a divisible heterogeneous good.

Main Question

How do we cut a cake fairly?

- We are interested not only in the *existence* of a (fair) division but also a *constructive procedure* (an algorithm) for finding it
  - discrete procedures
  - continuous moving knife procedures

Main Question

How do we cut a cake fairly?

- Different results known for 2,3,4,. . . cutters!

Main Question

How do we cut a cake fairly?

- Many ways to make this precise!

Main References

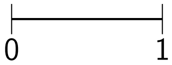
S. Brams and A. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. 1996.

J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair If You Can*. 1998.

J. Barbanel. *The Geometry of Efficient Fair Division*. 2005.

The Cake-Cutting Problem

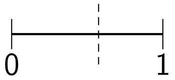
The cake is the unit interval $[0, 1]$



The Cake-Cutting Problem

The cake is the unit interval $[0, 1]$

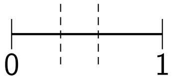
Only parallel, vertical cuts, perpendicular to the horizontal x-axis are made



The Cake-Cutting Problem

The cake is the unit interval $[0, 1]$

Only parallel, vertical cuts, perpendicular to the horizontal x-axis are made



The Cake-Cutting Problem

Each player i has a continuous value measure $v_i(x)$ on $[0, 1]$ such that

- $v_i(x) \geq 0$ for $x \in [0, 1]$
- v_i is finitely additive, non-atomic, and absolutely continuous measures
- the area under v_i on $[0, 1]$ is 1 (probability density function)

The Cake-Cutting Problem

Each player i has a continuous value measure $v_i(x)$ on $[0,1]$ such that

- $v_i(x) \geq 0$ for $x \in [0,1]$
- v_i is finitely additive, non-atomic, and absolutely continuous measures
- the area under v_i on $[0,1]$ is 1 (probability density function)

value of finite number of disjoint pieces equals the value of their union (hence, no subpieces have greater value than the larger piece containing them).

The Cake-Cutting Problem

Each player i has a continuous value measure $v_i(x)$ on $[0, 1]$ such that

- $v_i(x) \geq 0$ for $x \in [0, 1]$
- v_i is finitely additive, non-atomic, and absolutely continuous measures
- the area under v_i on $[0, 1]$ is 1 (probability density function)

a single cut (which defines the border of a piece) has no area and so has no value.

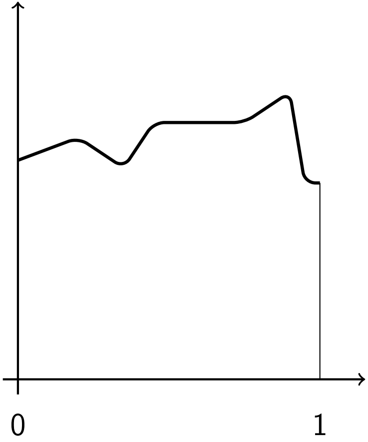
The Cake-Cutting Problem

Each player i has a continuous value measure $v_i(x)$ on $[0, 1]$ such that

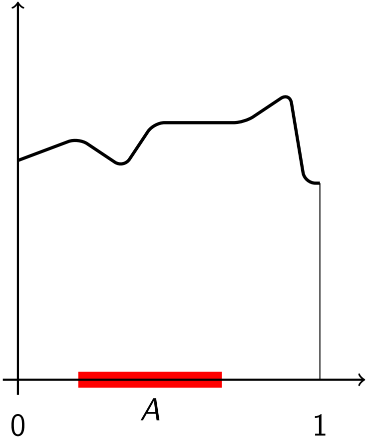
- $v_i(x) \geq 0$ for $x \in [0, 1]$
- v_i is finitely additive, non-atomic, and absolutely continuous measures
- the area under v_i on $[0, 1]$ is 1 (probability density function)

no portion of cake is of positive measure for one player and zero measure for another player.

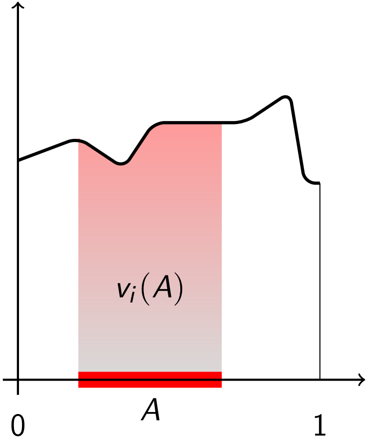
The Cake-Cutting Problem



The Cake-Cutting Problem



The Cake-Cutting Problem



Fairness

A division of a cake $[0, 1]$ for n players is a partition $(S_1, \ldots, S_n)$ (i.e., each $S_i \subseteq [0, 1]$, $\cup_i S_i = [0, 1]$ and $S_i \cap S_j = \emptyset$). We are typically interested in divisions where each S_i is **contiguous** (i.e., a subinterval of $[0, 1]$).

Fairness

A division of a cake $[0, 1]$ for n players is a partition $(S_1, \dots, S_n)$ (i.e., each $S_i \subseteq [0, 1]$, $\cup_i S_i = [0, 1]$ and $S_i \cap S_j = \emptyset$). We are typically interested in divisions where each S_i is **contiguous** (i.e., a subinterval of $[0, 1]$).

Fairness

A division of a cake $[0,1]$ for n players is a partition $(S_1, \ldots, S_n)$ (i.e., each $S_i \subseteq [0,1]$, $\cup_i S_i = [0,1]$ and $S_i \cap S_j = \emptyset$). We are typically interested in divisions where each S_i is **contiguous** (i.e., a subinterval of $[0,1]$).

A division $(S_1, \ldots, S_n)$ is

- **Fair (Proportional)**: for each i, $v_i(S_i) \geq \frac{1}{n}$

Fairness

A division of a cake $[0, 1]$ for n players is a partition $(S_1, \ldots, S_n)$ (i.e., each $S_i \subseteq [0, 1]$, $\cup_i S_i = [0, 1]$ and $S_i \cap S_j = \emptyset$). We are typically interested in divisions where each S_i is **contiguous** (i.e., a subinterval of $[0, 1]$).

A division $(S_1, \ldots, S_n)$ is

- **Fair (Proportional)**: for each i, $v_i(S_i) \geq \frac{1}{n}$
- **Envy-Free**: for each i, j, $v_i(S_i) \geq v_i(S_j)$

Fairness

A division of a cake $[0, 1]$ for n players is a partition $(S_1, \dots, S_n)$ (i.e., each $S_i \subseteq [0, 1]$, $\cup_i S_i = [0, 1]$ and $S_i \cap S_j = \emptyset$). We are typically interested in divisions where each S_i is **contiguous** (i.e., a subinterval of $[0, 1]$).

A division $(S_1, \dots, S_n)$ is

- **Fair (Proportional)**: for each i, $v_i(S_i) \geq \frac{1}{n}$
- **Envy-Free**: for each i, j, $v_i(S_i) \geq v_i(S_j)$
- **Equitable**: for each i, j, $v_i(S_i) = v_j(S_j)$

Fairness

A division of a cake $[0,1]$ for n players is a partition $(S_1, \ldots, S_n)$ (i.e., each $S_i \subseteq [0,1]$, $\cup_i S_i = [0,1]$ and $S_i \cap S_j = \emptyset$). We are typically interested in divisions where each S_i is **contiguous** (i.e., a subinterval of $[0,1]$).

A division $(S_1, \ldots, S_n)$ is

- **Fair (Proportional)**: for each i, $v_i(S_i) \geq \frac{1}{n}$
- **Envy-Free**: for each i, j, $v_i(S_i) \geq v_i(S_j)$
- **Equitable**: for each i, j, $v_i(S_i) = v_j(S_j)$
- **Efficient**: there is no other division $(T_1, \ldots, T_n)$ such that $v_i(T_i) \geq v_i(S_i)$ for all i and there is some j such that $v_j(T_j) > v_j(S_j)$.

Truthfulness

Some procedures ask players to represent their preferences.

This representation need not be "truthful"

Typically, it is assumed that agents will follow a maximin strategy (maximize the set of items that are guaranteed)

Cut-and-choose

Two Players

Procedure: one player cuts the cake into two portions and the other player chooses one of the portions.

Two Players

Procedure: one player cuts the cake into two portions and the other player chooses one of the portions.

Maximin strategy: Suppose that A is the cutter. If A has no information about the other player's valuation, then A should cut the cake at some point x so that the value of the portion to the left of x is equal to the value of the portion to the right.

Two Players

Procedure: one player cuts the cake into two portions and the other player chooses one of the portions.

Maximin strategy: Suppose that A is the cutter. If A has no information about the other player's valuation, then A should cut the cake at some point x so that the value of the portion to the left of x is equal to the value of the portion to the right.

This strategy creates an **envy-free** and **efficient** allocation, but it is not necessarily **equitable**.

Example

Suppose that the cake is half chocolate and have vanilla.
Ann values the vanilla half twice as much as the chocolate half:

$$v_A(x) = \begin{cases} 4/3 & x \in [0, 1/2] \\ 2/3 & x \in (1/2, 1] \end{cases}$$

Bob values both sides equally:

$$v_B(x) = \begin{cases} 1 & x \in [0, 1/2] \\ 1 & x \in (1/2, 1] \end{cases}$$

Where should A cut the cake?

Example

$$v_A(x) = \begin{cases} 4/3 & x \in [0, 1/2] \\ 2/3 & x \in (1/2, 1] \end{cases}$$

$$v_B(x) = \begin{cases} 1 & x \in [0, 1/2] \\ 1 & x \in (1/2, 1] \end{cases}$$

A should cut the cake at $x = 3/8$:

$$(4/3)(x - 0) = 4/3(1/2 - x) + 2/3(1 - 1/2)$$

Example

$$v_A(x) = \begin{cases} 4/3 & x \in [0, 1/2] \\ 2/3 & x \in (1/2, 1] \end{cases}$$

$$v_B(x) = \begin{cases} 1 & x \in [0, 1/2] \\ 1 & x \in (1/2, 1] \end{cases}$$

A should cut the cake at $x = 3/8$:

$$(4/3)(x - 0) = 4/3(1/2 - x) + 2/3(1 - 1/2)$$

Note that the portions are not equitable (B receive $5/8$ according to his valuation)

Cut and Choose is not Equitable

Suppose A values the vanilla half twice as much as the chocolate half:

$$v_A(x) = \begin{cases} 4/3 & x \in [0, 1/2] \\ 2/3 & x \in (1/2, 1] \end{cases} \qquad v_B(x) = \begin{cases} 1 & x \in [0, 1/2] \\ 1 & x \in (1/2, 1] \end{cases}$$

A should cut the cake at $x = 3/8$:

$$(4/3)(x - 0) = 4/3(1/2 - x) + 2/3(1 - 1/2)$$

The portions are not equitable: B receive $5/8$ according to his valuation.